# 4-3. 連成振動子

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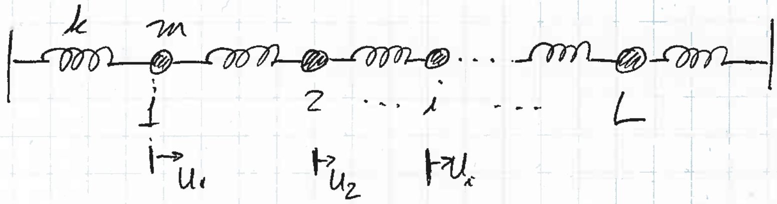



$$\mathcal{H} = \frac{1}{2m}\sum_{i=1} p_i^2 + \frac{k}{2}\sum (u_i - u_{i+1})^2 \quad ; \quad u_0 = u_{L+1} = 0$$

基準振動

$$\mathcal{H} \to \mathcal{H} = \frac{1}{2m}\sum_{n=1}^{L} p_n^2 + \frac{1}{2}\sum_{n=1}^{L} \lambda_n \xi_n^2$$ … N個の独立な調和振動子の和

(可解)

$$\lambda_n = 2k\left(1 - \cos\frac{n\pi}{L+1}\right);\ n = 1, \dots, L \ ; \ q_n = \frac{n\pi}{L+1}$$

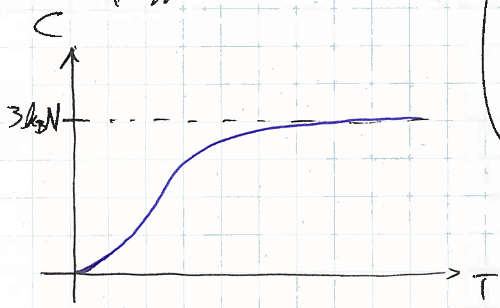
# 4-4. 固体の比熱

$$\mathcal{H} = \frac{p^2}{2m} + \frac{k}{2}x^2$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$E = \frac{\hbar\omega}{2} + \frac{\hbar\omega e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} \xrightarrow[\substack{\beta \to 0 \\ \hbar \to 0}]{} \frac{\hbar\omega}{2} + \underline{k_B T}$$

$$C = k_B(\beta\hbar\omega)^2 \frac{e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega} - 1)^2} = k_B(\beta\hbar\omega)^2 \frac{e^{-\beta\hbar\omega}}{(1 - e^{-\beta\hbar\omega})^2} \longrightarrow \underline{k_B}$$



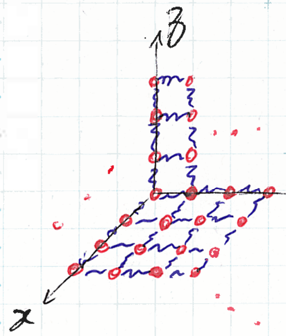

連成振動子でのω

$$\omega_n = \sqrt{\frac{2k}{m}(1 - \cos q_n)}$$

$$\langle E \rangle = \sum_{n=1}^{L}\left(\underbrace{\frac{\hbar\omega_n}{2}}_{\text{書かない(?)}} + \frac{\hbar\omega_n e^{-\beta\hbar\omega_n}}{1 - e^{-\beta\hbar\omega_n}}\right)$$

# 4-4-1. デバイ比熱

$$C=\sum_{n_x=1}^{L}\sum_{n_y=1}^{L}\sum_{n_z=1}^{L} k_B(\beta\hbar\omega_{\vec{n}})^2\frac{e^{-\beta\hbar\omega_{\vec{n}}}}{(1-e^{-\beta\hbar\omega_{\vec{n}}})^2}$$




$$\lambda_{\underbrace{(n_x,n_y,n_z)}_{\vec{n}}}=k\left(6-2\cos\left(\frac{n_x\pi}{L+1}\right)-2\cos\left(\frac{n_y\pi}{L+1}\right)-2\cos\left(\frac{n_z\pi}{L+1}\right)\right)$$

$$=k\left(6-2\cos q_x-2\cos q_y-2\cos q_z\right)$$

$$\underset{q\to 0}{\simeq} kq^2 \;;\quad q^2=\sqrt{q_x^2+q_y^2+q_z^2}$$

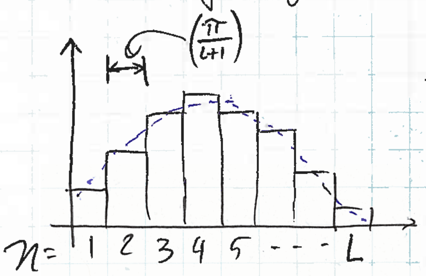
$$\omega_{\vec{q}}=\sqrt{\frac{k}{m}q^2}=cq$$ 波の速さ

$$C=\sum_{n_x=1}^{L}\sum_{n_y=1}^{L}\sum_{n_z=1}^{L}C_{\vec{n}}=\left(\frac{L+1}{\pi}\right)^3\int_0^{\pi}dq_x\int_0^{\pi}dq_y\int_0^{\pi}dq_z\,C_{\vec{q}}\simeq\frac{N}{\pi^3}\int_0^{q_{max}}4\pi q\,dq\times\frac{C_q}{8}$$

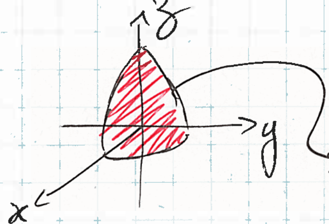
$$\sum_{n_x=1}^{L}=\frac{L+1}{\pi}\int_0^{\pi}dq_x\;;\quad(L+1)^3\simeq N$$

($C_{\vec{q}}=C_q$ のとき)

$q$ 大からの寄与は小さい




$$q_n=\frac{\pi}{L+1}n$$




$$4\pi q^2dq\rightarrow\frac{1}{8}\times4\pi q^2dq.$$

$$\Rightarrow C\equiv\int_0^{q_{max}}D(q)\,C_q\,dq$$

$$D(q)\,dq=\frac{N}{\pi^3}4\pi q^2\frac{dq}{8}=\frac{N}{2\pi^2}q^2dq$$

状態密度, Density of states

$$C = \int_0^{q_{max}} dq \frac{Nq^2}{2\pi^2} k_B \frac{(\beta\hbar\omega)^2 e^{-\beta\hbar\omega}}{(1-e^{-\beta\hbar\omega})^2}, \quad \omega = cq$$

$$= \frac{N}{2\pi^2}\frac{k_B}{c^3}\int_0^{q_{max}c=\omega_{max}} d\omega\, \omega^2 (\beta\hbar\omega)^2 \frac{e^{-\beta\hbar\omega}}{(1-e^{-\beta\hbar\omega})^2}$$

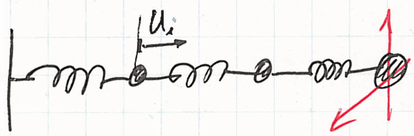
$$x=\beta\hbar\omega \quad = \frac{Nk_B}{2\pi^2}\frac{1}{c^3}\frac{1}{(\beta\hbar)^3}\int_0^{x_{max}=\beta\hbar\omega_{max}} dx \frac{x^4 e^{-x}}{(1-e^{-x})^2}$$

$(T \to 0)$

$\beta \to \infty,\ x_{max} \to \infty$

$$\longrightarrow \frac{Nk_B}{2\pi^2}\frac{1}{c^3}\frac{(k_BT)^3}{\hbar^3}\underbrace{\int_0^\infty \frac{x^4 e^{-x}}{(1-e^{-x})^2}dx}_{\frac{4\pi^4}{15}} \propto T^3 \ (T\to 0)$$

$$C \longrightarrow \frac{2\pi^2}{15c^3} N k_B \left(\frac{k_BT}{\hbar}\right)^3$$




横波 … $C_t$

タテ波 … $C_l$

$$\frac{1}{C_l^3} + \frac{2}{C_t^3} = \frac{3}{C^3}$$

$$T \to \infty \quad \int_0^{q_{max}} dq \frac{Nq^2}{2\pi^2} k_B = N k_B$$

$$\int_0^{q_{max}} q^2 dq = 2\pi^2 \Rightarrow q_{max} = (6\pi^2)^{1/3} \rightsquigarrow x_{max} = \beta\hbar c (6\pi^2)^{1/3}$$

$$\boxed{\Theta_D = \frac{\hbar\omega_{max}}{k} = \frac{\hbar}{k}(6c^3\pi^2)^{1/3}}$$ デバイ温度

$$\Longrightarrow C = \frac{12\pi^4}{5} N k_B \left(\frac{T}{\Theta_D}\right)^3$$

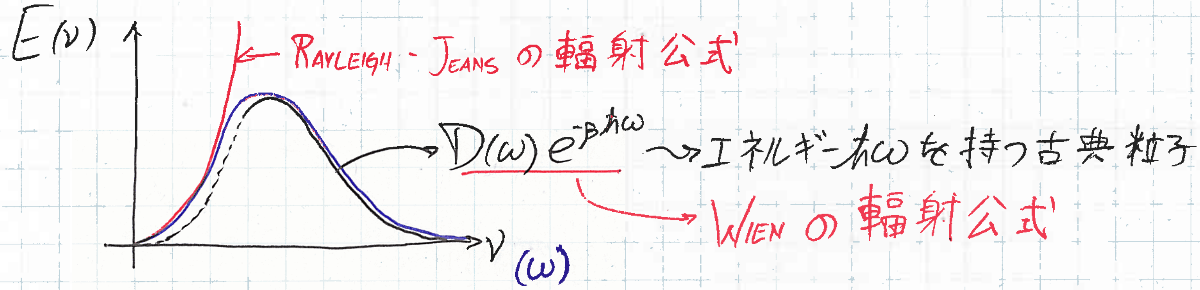
デバイ比熱

$q_n = \frac{\pi}{L+1} n \rightsquigarrow$ 周基境界条件のとき?

# 4-5. 黒体輻射

## 電磁波のスペクトル

$E(\nu)d\nu$　$E(\nu)$: 振動数 $2\pi\nu=\omega$ をもつ電磁波がもっているエネルギー




$$
\begin{cases}
\mathrm{div}\,\vec{D} = 0 \\
\mathrm{div}\,\vec{B} = 0 \\
\mathrm{rot}\,\vec{H} = \dfrac{\partial \vec{D}}{\partial t} \\
\mathrm{rot}\,\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}
\end{cases}
$$

$$
\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\frac{\partial^2}{\partial t^2}\vec{E}\mu_0\varepsilon_0
$$

$$
\parallel
$$

$$
\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{=0}) - \nabla^2\vec{E}
$$

$$
\begin{cases}
\nabla^2\vec{E} = \varepsilon_0\mu_0 \dfrac{\partial^2}{\partial t^2}\vec{E} \\
\nabla^2\vec{B} = \varepsilon_0\mu_0 \dfrac{\partial^2}{\partial t^2}\vec{B}
\end{cases}
$$

自由な電磁場は無限個の調和振動子の系とみなせる。

ベクトルポテンシャル $\vec{A}$

$\vec{E} = -\dfrac{\partial \vec{A}}{\partial t}$　$\vec{B} = \mathrm{rot}\,\vec{A}$　($\mathrm{div}\,\vec{A} = 0$ とする。)

$\left(\nabla^2\vec{A} = \varepsilon_0\mu_0 \dfrac{\partial^2}{\partial t^2}\vec{A}\right)$　$\omega = ck$ → 光速

$H_\lambda = \frac{1}{2}(p_\lambda^2 + k_\lambda^2 q_\lambda^2)$ … $\lambda$ 番目のノーマルモード.

$$
E(\omega)d\omega = 2D(\omega)d\omega\cdot E_\omega = \frac{V}{\pi^2c^3}\omega^2 d\omega(k_BT)
$$

2 → 偏光　$k_BT = E_\omega$

$$
= 2D(\omega)d\omega\left(\frac{\hbar\omega e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}\right)
$$

$$
= \frac{V}{\pi^2c^3}\omega^2\left( \quad 〃 \quad \right)d\omega
$$

→ Planck の輻射公式