理想フェルミ気体

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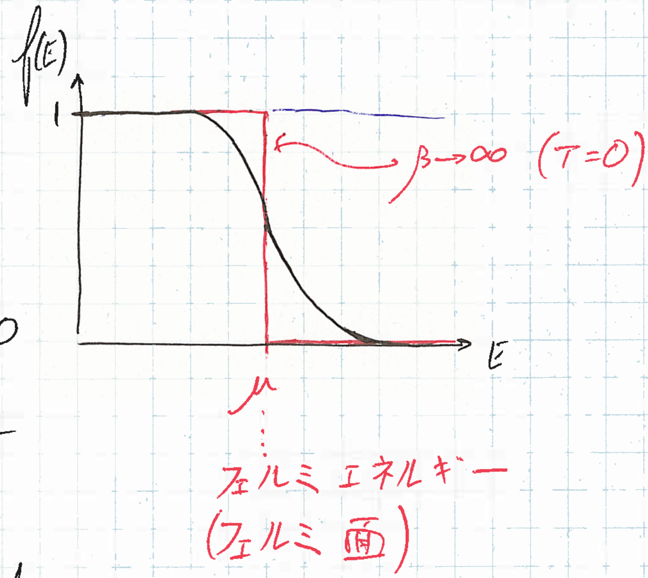


$$f(E) = \langle N_k \rangle = \frac{1}{e^{\beta(E_k-\mu)}+1}$$

$$\frac{\hbar^2 k^2}{2m} = E$$

$(\beta \to 大)$

$E_k > \mu \to f(E) \sim 0$

$E_k < \mu \to f(E) \sim 1$

与えられた個数　　期待値

$$\underbrace{N = \sum_k \langle N_k \rangle}_{\mu を決める条件} = \frac{L^3}{(2\pi)^3}\int_0^\infty 4\pi k^2 \frac{dk}{e^{\beta\left(\frac{\hbar^2k^2}{2m}-\mu\right)}+1}$$

$T=0$.

$$N = \frac{V}{(2\pi)^3}\int_0^{k_0} 4\pi k^2 dk = \frac{V}{(2\pi)^3}\frac{4\pi}{3}k_0^3 \rightsquigarrow k_0 = \left(\frac{N}{V}\frac{(2\pi)^3 \times 3}{4\pi}\right)^{1/3}$$

$$k_0 = \sqrt{\frac{2m\mu}{\hbar^2}} \Longrightarrow \mu_0 = \frac{\hbar^2}{2m}\left(\frac{N}{V}\frac{3}{4\pi}\right)^{2/3}$$ … $T=0$ のフェルミエネルギー.

$$E = \frac{L^3}{(2\pi)^3}\int_0^\infty 4\pi k^2 dk \frac{\frac{\hbar^2k^2}{2m}}{e^{\beta\left(\frac{\hbar^2k^2}{2m}-\mu\right)}+1}$$

$$E_0 = \frac{V}{(2\pi)^3}\int_0^{k_0}\frac{\hbar^2}{2m}4\pi k^4 dk = \frac{V}{(2\pi)^3}\frac{\hbar^2 4\pi}{2m}\frac{k_0^5}{5} = \frac{3}{5}N\mu_0$$

$A = \sum_k A(k)$ とする。 $\left(E = \sum_k \frac{\hbar^2}{2m}k^2\right)$

$$A = \frac{V}{(2\pi)^3}\int_0^\infty 4\pi k^2 dk\, A(k) f(k) \quad \cdots (3D).$$

$$= \int_0^\infty \underbrace{D(k)}_{状態密度}\underbrace{A(k)}_{物理量}\underbrace{f(k)}_{フェルミ分布}dk \;;\; f(k) = \frac{1}{e^{\beta(E(k)-\mu)}+1} \;;\; D(k)dk = \frac{V}{(2\pi)^3}4\pi k^2 dk$$

$$E = \frac{\hbar^2}{2m}k^2 \ ; \ dE = \frac{\hbar^2}{2m}2k\,dk = \frac{\hbar^2}{m}k\,dk$$

$$4\pi k^2 dk = 4\pi\left(\frac{2mE}{\hbar^2}\right)\frac{m}{\hbar^2}\left(\frac{2mE}{\hbar^2}\right)^{-1/2} dE = 4\pi\frac{m}{\hbar^2}\left(\frac{2mE}{\hbar^2}\right)^{1/2} dE$$

$\Rightarrow D(E)\,dE = \frac{V}{(2\pi)^3}\frac{4\pi m}{\hbar^3}\sqrt{2mE}\,dE$ ‥ 状態密度.（エネルギー）

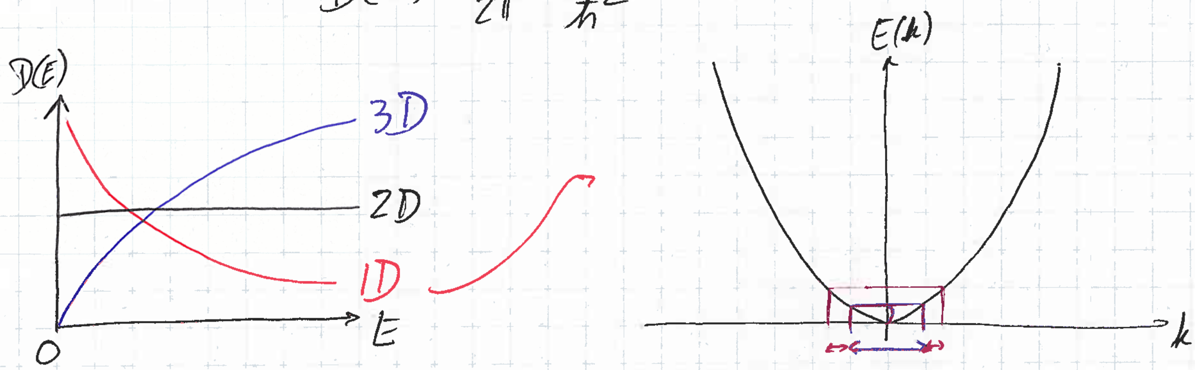
$$\langle A\rangle = \frac{V}{(2\pi)^3}\int_0^\infty A(\varepsilon)\underbrace{f(\varepsilon)}_{\frac{1}{e^{\beta(\varepsilon-\mu)}+1}}\underbrace{\frac{4\pi m}{\hbar^2}\left(\frac{2m\varepsilon}{\hbar^2}\right)^{1/2}}_{D(\varepsilon)} d\varepsilon$$

1次元　$D(k)\,dk = 2\frac{L}{2\pi}dk = 2\frac{L}{2\pi}\frac{m}{\hbar^2}\left(\frac{\hbar^2}{2mE}\right)^{+\frac{1}{2}} dE$

$\sum_{k_x} \to \int\frac{L}{2\pi}dk$　　$D(\varepsilon) = \frac{L}{\pi}\sqrt{\frac{m}{2\hbar^2}}\frac{1}{\sqrt{\varepsilon}}$

2次元　$D(k)\,dk = \left(\frac{L}{2\pi}\right)^2 2\pi k\,dk = \left(\frac{L}{2\pi}\right)^2 2\pi\frac{m}{\hbar^2}dE$

$$D(\varepsilon) = \frac{L^2}{2\pi}\frac{m}{\hbar^2}$$



$$\langle A \rangle = \int_0^\infty A(\varepsilon) D(\varepsilon) f(\varepsilon) d\varepsilon$$

$T \ll 1$ 低温での近似関係(ゾンマーフェルトの関係)

$$\langle A \rangle = \int_0^\mu D(\varepsilon) A(\varepsilon) d\varepsilon + \frac{\pi^2}{6}(k_B T)^2 2\pi V \left(\frac{2m}{h^2}\right)^{3/2} \cdot \frac{d}{d\mu}\left(\sqrt{\mu} A(\mu)\right) + \cdots$$

低温での性質

$$\langle N \rangle = \int_0^\mu D(\varepsilon) d\varepsilon + \frac{\pi^2}{6}(k_B T)^2 2\pi V \left(\frac{2m}{h^2}\right)^{3/2} \frac{d}{d\mu}\left(\sqrt{\mu}\right)$$

$A = N$

$A(E) = 1$

$a(\varepsilon) = D(\varepsilon)$

$$= 2\pi V \left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \left(\frac{2}{3}\mu^{3/2} + \frac{\pi^2}{12}(k_B T)^2 \frac{1}{\sqrt{\mu}}\right)$$

$$= 2\pi V \left(\frac{2m}{h^2}\right)^{3/2} \frac{2}{3}\mu_0^{3/2}$$

$$\underline{\mu_0^{3/2} = \mu^{3/2}}\left[1 + \frac{\pi^2}{8}\left(\frac{k_B T}{\mu}\right)^2\right] \Rightarrow \mu = \mu_0 \left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\mu_0}\right)^2 + \cdots\right].$$

$$\langle E \rangle = \int_0^\mu \varepsilon D(\varepsilon) d\varepsilon + \frac{\pi^2}{6}(k_B T)^2 2\pi V \left(\frac{2m}{h^2}\right)^{3/2} \frac{d}{d\mu}\left(\mu\sqrt{\mu}\right)$$

$A = E$

$A(E) = E$

$a(\varepsilon) = \varepsilon D(\varepsilon)$

$$= 2\pi V \left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \left(\frac{2}{5}\mu^{\frac{5}{2}} + \frac{\pi^2}{4}(k_B T)^2 \sqrt{\mu}\right)$$

μを代入, $O(T^2)$まで.

$$= E_0 \left(1 + \frac{5\pi^2}{12}\left(\frac{k_B T}{\mu_0}\right)^2\right)$$

$$C = \frac{d\langle E \rangle}{dT} = E_0 \frac{5\pi^2}{6}\left(\frac{k_B T}{\mu_0}\right)^2 = \frac{\pi^2}{2}\frac{k_B^2 T}{\mu_0} N$$

電子の場合 $N \longrightarrow$ ↑スピン $N/2$, ↓スピン $N/2$

$$\Rightarrow \mu_0 = \frac{h^2}{2m}\left(\frac{N}{V}\frac{3}{8\pi}\right)^{2/3}, \quad C = \frac{\pi^2}{2}\frac{k_B^2 T}{\mu_0}\left(\frac{N}{2} + \frac{N}{2}\right)$$

圧力

$$\Xi = \sum_{\text{ALL STATES IN } N} e^{-\beta(E_i(N)-\mu N)} \rightarrow k_B T \ln \Xi = PV$$

N

$$\Xi = \sum_{i,N} e^{-\beta(E_i - \mu N)} = \sum_{E,N} D(E) e^{-\beta(E-\mu N)} = \sum_{E,N} e^{\frac{S(E)}{k_B} - \frac{E-\mu N}{k_B T}}$$

$$\simeq e^{\frac{1}{k_B T}(S(E^*)T - E^* + \mu N^*)}$$

$E=E^*$, $N=N^*$ の所の寄与で決まる。
(鞍点法)

$$= e^{\frac{PV}{k_B T}} \Longrightarrow \boxed{\ln \Xi = \beta PV}$$

$E = TS - PV + \mu N$

$$PV = k_B T \ln \Xi \; ; \; \Xi = \prod_{k} \left(1 + e^{-\beta(E_k - \mu)}\right)$$

$$\Rightarrow PV = k_B T \sum_{k} \ln\left(1 + e^{-\beta(E_k-\mu)}\right) = k_B T \frac{V}{(2\pi)^3} \int_0^\infty 4\pi k^2 dk \ln\left(1 + e^{\beta(E_k-\mu)}\right)$$

$$= k_B T \frac{V}{(2\pi)^3} \int_0^\infty 4\pi \frac{k^3}{3} \underbrace{\frac{e^{-\beta(E_k-\mu)}}{1+e^{-\beta(E_k-\mu)}}}_{f(E)} \underbrace{\frac{dE_k}{dk}}_{\left(-2\frac{\beta\hbar^2}{2m}k\right)} dk = \frac{2}{3} E_0$$